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A NEW FRACTIONAL CURVATURE OF CURVES USING THE CAPUTO'S FRACTIONAL DERIVATIVE

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Abstract. In this work, the authors presented a new definition of fractional curvature of plane curves when the fractional derivative is in the Caputo sense. This new curvature is based on the behavior of the orthogonal projection of the integer derivative of the fractional derivative vector of the curve, on the vector normal to the curve. The goal of our research is to show that our fractional curvature belongs to the intrinsic geometry of the curve; since it is invariant under isometries.

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1 Introduction

Fractional calculus is a branch of mathematics that has acquired great importance today, since it allows the study of various problems in different areas of science, by using derivation and integration of non-integer order; as occurs in differential equations (Yusubov , 2020; Manafian & Allahverdiyera, 2021); viscoelasticity (Bagley & Torvik, 2083; Koeller, 1984; Yajima & Nagahama, 2010; Atanackovic & Stankovic, 2002; Beyer & Kempfle, 1995), medicine (Dokuyucu et al., 2018; Veeresha et al., 2019), dynamic systems Grigorenko & Grigorenko (2003); Yajima & Nagahama (2018), mechanics Drapaca & Sivaloganathan (2012), hydrodynamics Balankin & Elizarrataz (2012), etc.

Currently there are many definitions of fractional derivative, such as the Riemann-Liouville fractional derivative, Caputo, Marchand, Hadamard, etc. Bonilla (2003).

In geometry, fractional calculus is also used in the study of the geometric properties of curves (Aydin et al., 2021; Gozutok et al., 2019; Yajima et al., 2018; Aydin et al., 2021), surfaces (Lazopoulos & Lazopoulos, 2016; Yajima & Yamasaki, 2012), and Riemannian manifolds (Cottrill-Shepherd& Naber, 2001; Jumarie, 2013; Calcagni, 2012).

Caputo's fractional derivative, Caputo (1967), in geometry is also used, since the fractional derivative of the constant function is zero.

In Yajima et al. (2018), was forced to make a simplification of the fractional derivative of the composite function; since the formula for the fractional derivative of the composite function involves a series, which made it difficult to apply fractional calculus to the study of geometry.

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Using the fractional derivative of Caputo, in this paper a new definition of Fractional Curvature of plane curves is given, different from the approach of Yajima et al. (2018), which makes use of integration for its calculation (58), which does not occur with the new fractional curvature.

The goal of our research is to show that our fractional curvature belongs to the intrinsic geometry of the curve; since it is invariant under isometries. Furthermore, 1-dimensional Euclidean spaces are characterized as those spaces whose fractional curvature is zero at all points.

This paper is organized as follows: in section 2 results on differential geometry of curves (Do Carmo, 1976; Tenenblat, 2008), and fractional Caputo derivative Caputo (1967) are given. In section 3, fractional curvature is defined and the theory is developed. Examples in mathematics and physics are given in section 4.

2 Preliminaries

Definition 1. *(Tenenblat, 2008). A parameterized curve differentiable in $\mathbb{R}^n$ is a mapping $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n$ differentiable of class C^∞, on an open interval $I \subset \mathbb{R}$. The variable t is called a parameter, and the subset of $\mathbb{R}^n$ formed by the points $\alpha(t)$, is called the trace of the curve.*

Definition 2. *(Tenenblat, 2008). Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n$ a differentiable parameterized curve. The vector*

$$\alpha'(t) = \left(x_1'(t), x_2'(t), \ldots, x_n'(t)\right),$$

is called the Tangent Vector of α at t.

The curve $\alpha(t)$ is called Regular if

$$\alpha'(t) \neq 0, \quad \forall\, t \in I.$$

Furthermore, if α is a regular curve, it can be reparametrized by the arc length parameter s, where

$$s = s(t) = \int_{t_0}^{t} \|\alpha'(\tau)\| d\tau. \tag{1}$$

If α is parameterized by arc length, then $\|\alpha'(s)\| = 1, \quad \forall s \in I$.

2.1 Plane curves

Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parametrized by arc length. The Frenet-Serret frame of the plane curve $\alpha(s) = (x(s), y(s))$ is given by the orthonormal basis $\{t(s), n(s)\}, \forall s \in I$, where

$$t(s) = \alpha'(s) = \left(x'(s), y'(s)\right), \tag{2}$$
$$n(s) = \left(-y'(s), x'(s)\right). \tag{3}$$

The Frenet-Serret equations of the curve (Do Carmo, 1976; Tenenblat, 2008), are

$$t'(s) = k(s)n(s), \tag{4}$$
$$n'(s) = -k(s)t(s). \tag{5}$$

where $k(s)$ is the integer or classical curvature of the curve $\alpha(s)$ at point s, and is determined by

$$k(s) = \langle t'(s), n(s)\rangle, \quad \forall s \in I. \tag{6}$$

2.2 Caputo Fractional Derivative

Let $f : [a, b] \to \mathbb{R}$ be a function of class $C^1([a, b])$. The Caputo Fractional Derivative of order λ is defined by (Bonilla, 2003; Caputo, 1967)

$$ {}^cD^\lambda f(t) = \frac{1}{\Gamma(1-\lambda)} \int_a^t \frac{1}{(t-u)^\lambda} f'(u)du, \tag{7}$$

where $\lambda \in \mathbb{R}$, $0 < \lambda < 1$, and Γ is Euler's gamma function.

In this paper we will use the Caputo fractional derivative. Furthermore, our analysis is based on this property

$$\lim_{\lambda \to 1} {}^cD^\lambda f(t) = f'(t), \qquad \forall\, t \in [a, b]. \tag{8}$$

Therefore, for λ close to 1, the properties of the integer derivative can be approximated by the Caputo's fractional derivative.

3 Fractional Curvature

In this section we give a new definition of the fractional curvature that differs from the approach given by Yajima et al. (2018).

Definition 3. *Let $\alpha : [a, b] \to \mathbb{R}^2$ be a regular curve parametrized by arc length s. The Fractional Derivative vector of order λ of α at s, is given by*

$$ {}^cD^\lambda \alpha(s) = \left({}^cD^\lambda x(s), {}^cD^\lambda y(s) \right). \tag{9}$$

In the Frenet-Serret frame, we have:

$$ {}^cD^\lambda \alpha(s) = a^\lambda(s)t(s) + b^\lambda(s)n(s), \qquad \forall s \in [a, b]. \tag{10}$$

Definition 4. *Let $\alpha : [a, b] \to \mathbb{R}^2$ be a regular curve parametrized by arc length, ${}^cD^\lambda\alpha(s)$ the fractional derivative vector of order λ. The Fractional Curvature of order λ , $0 < \lambda < 1$, of the curve α at point s, is given by*

$$k^\lambda(s) = a^\lambda(s)k(s) + \frac{d}{ds}\left(b^\lambda(s)\right), \qquad \forall s \in [a, b], \tag{11}$$

where $k(s)$ is the integer curvature of α at point s.

Theorem 1. *Let $\alpha : [a, b] \to \mathbb{R}^2$ be a regular curve parametrized by arc length. Then*

$$\begin{aligned} Proj_{n(s)} \frac{d}{ds}\left({}^cD^\lambda\alpha(s)\right) &= k^\lambda(s)n(s), & (12)\\ k^\lambda(s) &= \langle Proj_{n(s)} \frac{d}{ds}\left({}^cD^\lambda\alpha(s)\right), n(s)\rangle = \langle \frac{d}{ds}\left({}^cD^\lambda\alpha(s)\right), n(s)\rangle. & (13)\end{aligned}$$

Proof. Finding the integer derivative in (10), we have

$$\frac{d}{ds}\left({}^cD^\lambda\alpha(s)\right) = \frac{d}{ds}\left(a^\lambda(s)\right)t(s) + a^\lambda(s)t'(s) + \frac{d}{ds}\left(b^\lambda(s)\right)n(s) + b^\lambda(s)n'(s). \tag{14}$$

Substituting (4) and (5) into (14), we obtain

$$\frac{d}{ds}\left({}^cD^\lambda\alpha(s)\right) = \left(\frac{d}{ds}\left(a^\lambda(s)\right) - b^\lambda(s)k(s)\right)t(s) + \left(a^\lambda(s)k(s) + \frac{d}{ds}\left(b^\lambda(s)\right)\right)n(s). \tag{15}$$

Then, the orthogonal projection is given by

$$\begin{aligned} Proj_{n(s)}\frac{d}{ds}\left({}^cD^\lambda\alpha(s)\right) &= k^\lambda(s)n(s), \\ k^\lambda(s) &= \langle Proj_{n(s)}\frac{d}{ds}\left({}^cD^\lambda\alpha(s)\right), n(s)\rangle. \end{aligned}$$

□

From (15), we define the function $\bar{k}^\lambda$ by

$$\overline{k}^\lambda(s) := \frac{d}{ds}\left(a^\lambda(s)\right) - b^\lambda(s)k(s), \quad \forall s \in [a,b]. \tag{16}$$

Theorem 2. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parametrized by arc length. Then*

$$\lim_{\lambda\to 1} a^\lambda(s) = 1, \quad \forall s \in [a,b], \tag{17}$$

$$\lim_{\lambda\to 1} b^\lambda(s) = 0, \quad \forall s \in [a,b]. \tag{18}$$

Proof. By (8), we have

$$\lim_{\lambda\to 1} {}^cD^\lambda\alpha(s) = \alpha'(s) = t(s), \quad \forall s \in [a,b],$$

and by (10) we obtain the result. □

Theorem 3. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parameterized by arc length, Then*

$$a^\lambda(s) = x'(s){}^cD^\lambda x(s) + y'(s){}^cD^\lambda y(s), \tag{19}$$

$$b^\lambda(s) = x'(s){}^cD^\lambda y(s) - y'(s){}^cD^\lambda x(s). \tag{20}$$

Proof. By (9) and (10), we get

$$\begin{aligned} {}^cD^\lambda\alpha(s) &= \left({}^cD^\lambda x(s), {}^cD^\lambda y(s)\right) = a^\lambda(s)t(s) + b^\lambda(s)n(s) \\ &= a^\lambda(s)\left(x'(s), y'(s)\right) + b^\lambda(s)\left(-y'(s), x'(s)\right) \\ &= \left(a^\lambda(s)x'(s) - b^\lambda(s)y'(s), a^\lambda(s)y'(s) + b^\lambda(s)x'(s)\right). \end{aligned}$$

We obtain the system of equations:

$$\left\{ \begin{aligned} x'(s)a^\lambda(s) - y'(s)b^\lambda(s) &= {}^cD^\lambda x(s) \\ y'(s)a^\lambda(s) + x'(s)b^\lambda(s) &= {}^cD^\lambda y(s) \end{aligned} \right. . \tag{21}$$

Furthermore, the determinant of system (21) is non-zero:

$$\begin{vmatrix} x'(s) & -y'(s) \\ y'(s) & x'(s) \end{vmatrix} = 1, \quad \forall s \in [a,b].$$

By the Cramer's rule the result follows. □

Theorem 4. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parameterized by arc length. Then*

$$\lim_{\lambda\to 1} k^\lambda(s) = k(s), \quad \forall s \in [a,b]. \tag{22}$$

Proof. By (11) we have:

$$k^\lambda(s) = a^\lambda(s)k(s) + \frac{d}{ds}\left(b^\lambda(s)\right), \quad \forall s \in [a,b],$$

and by (17) and (18), we have

$$\lim_{\lambda\to 1} k^\lambda(s) = \lim_{\lambda\to 1} a^\lambda(s)k(s) + \frac{d}{ds}\left(\lim_{\lambda\to 1} b^\lambda(s)\right) = k(s), \quad \forall s \in [a,b].$$

This completes the proof. □

Theorem 5. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a curve parametrized by arc length. Then*

$$k^\lambda(s) = x'(s)\frac{d}{ds}\left({}^cD^\lambda y(s)\right) - y'(s)\frac{d}{ds}\left({}^cD^\lambda x(s)\right). \tag{23}$$

Proof. By (9), we have

$$\frac{d}{ds}\left({}^cD^\lambda \alpha(s)\right) = \left(\frac{d}{ds}\left({}^cD^\lambda x(s)\right), \frac{d}{ds}\left({}^cD^\lambda y(s)\right)\right).$$

And by (13):

$$k^\lambda(s) = \left\langle \frac{d}{ds}\left({}^cD^\lambda \alpha(s)\right), n(s)\right\rangle = \left\langle \frac{d}{ds}\left({}^cD^\lambda \alpha(s)\right), (-y'(s), x'(s))\right\rangle.$$

Therefore: $k^\lambda(s) = x'(s)\frac{d}{ds}\left({}^cD^\lambda y(s)\right) - y'(s)\frac{d}{ds}\left({}^cD^\lambda x(s)\right)$. □

Theorem 6. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parameterized by arc length. Then*

$$\left\|\frac{d}{ds}\left({}^cD^\lambda \alpha(s)\right)\right\|^2 = \left(\bar{k}^\lambda(s)\right)^2 + \left(k^\lambda(s)\right)^2, \quad \forall s \in [a,b]. \tag{24}$$

Proof. By considering (11), (15) and (16), we obtain the equality (24). □

Theorem 7. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parameterized by arc length. Then*

$$2\left(a^\lambda(s)\bar{k}^\lambda(s) + b^\lambda(s)k^\lambda(s)\right) = \frac{d}{ds}\left\|{}^cD^\lambda \alpha(s)\right\|^2. \tag{25}$$

Proof. We multiply (11) by $b^\lambda(s)$ and (16) by $a^\lambda(s)$ and adding the equalities, we obtain:

$$\begin{aligned} a^\lambda(s)\bar{k}^\lambda(s) + b^\lambda(s)k^\lambda(s) &= a^\lambda(s)\frac{d}{ds}\left(a^\lambda(s)\right) + b^\lambda(s)\frac{d}{ds}\left(b^\lambda(s)\right) \\ &= \frac{1}{2}\frac{d}{ds}\left(\left(a^\lambda(s)\right)^2 + \left(b^\lambda(s)\right)^2\right) \\ &= \frac{1}{2}\frac{d}{ds}\left(\left\|{}^cD^\lambda \alpha(s)\right\|^2\right). \end{aligned}$$

Therefore

$$\frac{d}{ds}\left(\left\|{}^cD^\lambda \alpha(s)\right\|^2\right) = 2\left(a^\lambda(s)\bar{k}^\lambda(s) + b^\lambda(s)k^\lambda(s)\right).$$

This completes the proof. □

Theorem 8. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parameterized by arc length. Then, there are functions $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$, such that*

$$f(s)k^\lambda(s) = k(s), \qquad \forall s \in [a,b], \tag{26}$$

$$g(s)\bar{k}^\lambda(s) = -k(s), \qquad \forall s \in [a,b]. \tag{27}$$

Proof. By (15), we have

$$\frac{d}{ds}\left({}^{c}D^{\lambda}\alpha(s)\right) = \bar{k}^{\lambda}(s)t(s) + k^{\lambda}(s)n(s), \quad \forall s \in [a,b].$$

Then:

$$Proj_{n(s)}\left(\frac{d}{ds}\left({}^{c}D^{\lambda}\alpha(s)\right)\right) \text{ is parallel to } \ t'(s), \quad \forall s \in [a,b],$$

$$k^{\lambda}(s)n(s) \text{ is parallel to } \ t'(s) = k(s)n(s), \quad \forall s \in [a,b].$$

Then, there is a function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ such that:

$$f(s)k^{\lambda}(s) = k(s), \quad \forall s \in [a,b].$$

Analogously, we have

$$Proj_{t(s)}\left(\frac{d}{ds}\left({}^{c}D^{\lambda}\alpha(s)\right)\right) \text{ is parallel to } \ n'(s), \quad \forall s \in [a,b],$$

$$\bar{k}^{\lambda}(s)t(s) \text{ is parallel to } \ n'(s) = -k(s)t(s), \quad \forall s \in [a,b].$$

Then, there is a function $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$ such that

$$g(s)\bar{k}^{\lambda}(s) = -k(s), \quad \forall s \in [a,b].$$

This completes the proof. □

Corollary 1. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parameterized by arc length. Then, there are functions $f, g : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$, such that*

$$k(s) = \frac{1}{2}\left(f(s)k^{\lambda}(s) - g(s)\bar{k}^{\lambda}(s)\right), \quad \forall s \in [a,b], \tag{28}$$

$$f(s)k^{\lambda}(s) + g(s)\bar{k}^{\lambda}(s) = 0, \quad \forall s \in [a,b]. \tag{29}$$

Proof. By considering (26) and (27), we prove the result. □

The next theorem gives a characterization of 1-dimensional Euclidean spaces, through the fractional curvature.

Theorem 9. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parameterized by arc length. Then*

$$k^{\lambda}(s) = 0 \quad \text{if and only if} \quad k(s) = 0, \quad \forall s \in [a,b], \quad \forall \lambda \in <0,1> . \tag{30}$$

Proof. Suppose that $k^{\lambda}(s) = 0, \quad \forall s \in [a,b], \quad \forall \lambda \in <0,1>$, then by (26) we have

$$k(s) = f(s)k^{\lambda}(s) = 0, \quad \forall s \in [a,b].$$

Conversely, suppose that $k(s) = 0, \forall s \in [a,b]$. Then, by the Frenet-Serret equation (4), we have

$$\alpha(s) = (x_0 + sv_1, y_0 + sv_2) = (x(s), y(s)) \quad v = (v_1, v_2) \in \mathbb{R}^2, \quad \|v\| = 1.$$

The coordinate functions are: $x(s) = x_0 + sv_1$, $y(s) = y_0 + sv_2$.

Finding the integer derivative of the functions, we get

$$x'(s) = v_1, \qquad y'(s) = v_2, \quad \forall s \in [a,b]. \tag{31}$$

We find the Caputo derivative

$$^{c}D^{\lambda}x(s) = \frac{s^{1-\lambda}}{\Gamma(2-\lambda)}v_1 \quad , \quad ^{c}D^{\lambda}y(s) = \frac{s^{1-\lambda}}{\Gamma(2-\lambda)}v_2. \tag{32}$$

Using the equalities (31) and (32), we find a^{λ} and b^{λ} :

$$a^{\lambda}(s) = x'(s)^{c}D^{\lambda}x(s) + y'(s)^{c}D^{\lambda}y(s) = \frac{s^{1-\lambda}}{\Gamma(2-\lambda)}, \quad \forall s \in [a,b], \tag{33}$$

$$b^{\lambda}(s) = x'(s)^{c}D^{\lambda}y(s) - y'(s)^{c}D^{\lambda}x(s) = 0, \quad \forall s \in [a,b]. \tag{34}$$

Substituting (33) and (34) into (11), we obtain $k^{\lambda}(s) = 0, \forall s \in [a,b], \lambda \in <0,1>$. □

Theorem 10. *Let $\alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve parameterized by arc length, let $F : \mathbb{R}^2 \to \mathbb{R}^2$ an isometry and $\beta = F \circ \alpha : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$. Then*

$$^{c}D^{\lambda}\beta(s) = dF_{\alpha(s)}\left(^{c}D^{\lambda}\alpha(s)\right), \tag{35}$$

$$k^{\lambda}_{\beta}(s) = k^{\lambda}_{\alpha}(s), \tag{36}$$

where $k^{\lambda}_{\alpha}(s)$ and $k^{\lambda}_{\beta}(s)$ are the fractional curvatures of the curves α and β respectively; and $dF_{\alpha(s)} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the differential of F at $\alpha(s)$.

Proof. Let $\{t_{\alpha}(s), n_{\alpha}(s)\}$, $\{t_{\beta}(s), n_{\beta}(s)\}$ be the Frenet-Serret frames of the curves α and β respectively.

Since F is an isometry, then

$$t_{\beta}(s) = dF_{\alpha(s)}\left(t_{\alpha}(s)\right), \tag{37}$$

$$n_{\beta}(s) = dF_{\alpha(s)}\left(n_{\alpha}(s)\right), \tag{38}$$

$$k_{\beta}(s) = k_{\alpha}(s). \tag{39}$$

where $k_{\beta}(s)$ and $k_{\alpha}(s)$ denote the integer curvature of the curves α and β respectively.

Furthermore, we have

$$\begin{aligned} ^{c}D^{\lambda}\alpha(s) &= a^{\lambda}(s)t_{\alpha}(s) + b^{\lambda}(s)n_{\alpha}(s), \\ ^{c}D^{\lambda}\beta(s) &= A^{\lambda}(s)t_{\beta}(s) + B^{\lambda}(s)n_{\beta}(s). \end{aligned}$$

Since F is an isometry, there exist a translation T_p and an orthogonal transformation C, such that

$$F = T_p \circ C. \tag{40}$$

By (40), we have

$$\begin{aligned} \beta(s) &= \left(T_p \circ \alpha\right)(s) = \left(T_p \circ C\right)(\alpha(s)), \\ &= p + C(\alpha(s)) = p + C\left(x(s)e_1 + y(s)e_2\right) \\ &= p + x(s)C(e_1) + y(s)C(e_2). \end{aligned}$$

Then, we find the Caputo derivative

$$\begin{aligned} ^{c}D^{\lambda}\beta(s) &= {}^{c}D^{\lambda}(p) + \left(^{c}D^{\lambda}x(s)\right)C(e_1) + \left(^{c}D^{\lambda}y(s)\right)C(e_2), \\ &= C\left(^{c}D^{\lambda}x(s)e_1 + {}^{c}D^{\lambda}y(s)e_2\right) = C\left(^{c}D^{\lambda}\alpha(s)\right). \end{aligned}$$

Hence $^{c}D^{\lambda}\beta(s) = C\left(^{c}D^{\lambda}\alpha(s)\right) = dF_{\alpha(s)}\left(^{c}D^{\lambda}\alpha(s)\right)$.

Also, by (37) and (38), we have

$$\begin{aligned} A^\lambda(s)t_\beta(s)+B^\lambda(s)n_\beta(s) &= {}^cD^\lambda\beta(s)=dF_{\alpha(s)}\left(a^\lambda(s)t_\alpha(s)+b^\lambda(s)n_\alpha(s)\right)\\ &= a^\lambda(s)dF_{\alpha(s)}\left(t_\alpha(s)\right)+b^\lambda(s)dF_{\alpha(s)}\left(n_\alpha(s)\right)\\ &= a^\lambda(s)t_\beta(s)+b^\lambda(s)n_\beta(s). \end{aligned}$$

Then

$$\begin{aligned} A^\lambda(s) &= a^\lambda(s), \quad \forall s\in[a,b],\\ B^\lambda(s) &= b^\lambda(s), \quad \forall s\in[a,b]. \end{aligned}$$

Therefore, we get

$$k^\lambda_\beta(s)=A^\lambda(s)k_\beta(s)+\frac{d}{ds}\left(B^\lambda(s)\right)=a^\lambda(s)k_\alpha(s)+\frac{d}{ds}\left(b^\lambda(s)\right)=k^\lambda_\alpha(s), \quad \forall s\in[a,b].$$

□

Next, we will obtain the results for a regular curve with arbitrary parameter r.

Let $\alpha:[c,d]\subset\mathbb{R}\to\mathbb{R}^2$ a regular curve with arbitrary parameter r, $\{t(r),n(r)\}$ the Frenet-Serret frame of α at r, where

$$t(r)=\frac{\alpha'(r)}{\|\alpha'(r)\|}=\frac{(x'(r),y'(r))}{\|\alpha'(r)\|}, \qquad n(r)=\frac{(-y'(r),x'(r))}{\|\alpha'(r)\|}. \tag{41}$$

Furthermore, we have

$$\begin{aligned} t'(r) &= \|\alpha'(r)\|k(r)n(r), &(42)\\ n'(r) &= -\|\alpha'(r)\|k(r)t(r), &(43) \end{aligned}$$

where

$$k(r)=\frac{-x''(r)y'(r)+y''(r)x'(r)}{\|\alpha'(r)\|^3}. \tag{44}$$

is the integer curvature of α at r.

Therefore

$${}^cD^\lambda\alpha(r)=a^\lambda(r)t(r)+b^\lambda(r)n(r), \quad \forall r\in[c,d]. \tag{45}$$

By (8), we have the following result.

Theorem 11. *Let $\alpha:[c,d]\subset\mathbb{R}\to\mathbb{R}^2$ a regular curve with arbitrary parameter r. Then:*

$$\begin{aligned} \lim_{\lambda\to1}a^\lambda(r) &= \|\alpha'(r)\|, \quad \forall r\in[c,d], &(46)\\ \lim_{\lambda\to1}b^\lambda(r) &= 0, \quad \forall r\in[c,d]. &(47) \end{aligned}$$

Proof. The proof is immediate. □

Finding the integer derivative in (45):

$$\frac{d}{dr}\left({}^cD^\lambda\alpha(r)\right)=\frac{d}{dr}\left(a^\lambda(r)\right)t(r)+a^\lambda(r)t'(r)+\frac{d}{dr}\left(b^\lambda(r)\right)n(r)+b^\lambda(r)n'(r). \tag{48}$$

Using the equalities (42) and (43) in (48), we get

$$\frac{d}{dr}\left({}^cD^\lambda\alpha(r)\right)=\left[\frac{d}{dr}\left(a^\lambda(r)\right)-b^\lambda(r)\|\alpha'(r)\|k(r)\right]t(r)+\left[a^\lambda(r)\|\alpha'(r)\|k(r)+\frac{d}{dr}\left(b^\lambda(r)\right)\right]n(r). \tag{49}$$

From (49), we define the function $k_1^\lambda : [c,d] \to \mathbb{R}$, by

$$k_1^\lambda(r) = a^\lambda(r)\|\alpha'(r)\|k(r) + \frac{d}{dr}\left(b^\lambda(r)\right). \tag{50}$$

From (49), we have

$$k_1^\lambda(r) = \langle \frac{d}{dr}\left({}^cD^\lambda\alpha(r)\right), n(r)\rangle, \quad \forall r \in [c,d].$$

Theorem 12. *Let $\alpha : [c,d] \subset \mathbb{R} \to \mathbb{R}^2$ a regular curve with arbitrary parameter r. Then:*

$$k(r) = \lim_{\lambda\to 1} \frac{k_1^\lambda(r)}{\|\alpha'(r)\|^2}, \quad \forall r \in [c,d]. \tag{51}$$

Proof. By (50), we have

$$k_1^\lambda(r) = a^\lambda(r)\|\alpha'(r)\|k(r) + \frac{d}{dr}\left(b^\lambda(r)\right).$$

Then, finding the limit

$$\lim_{\lambda\to 1} k_1^\lambda(r) = \left(\lim_{\lambda\to 1} a^\lambda(r)\right)\|\alpha'(r)\|k(r) + \lim_{\lambda\to 1}\left(\frac{d}{dr}\left(b^\lambda(r)\right)\right).$$

By (46) and (47), we get

$$\lim_{\lambda\to 1} k_1^\lambda(r) = \|\alpha'(r)\|^2 k(r).$$

Therefore:

$$k(r) = \lim_{\lambda\to 1} \frac{k_1^\lambda(r)}{\|\alpha'(r)\|}, \quad \forall r \in [c,d].$$

□

From (51) we have the next definition.

Definition 5. *Let $\alpha : [c,d] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve with arbitrary parameter r, ${}^cD^\lambda\alpha(r)$ the fractional derivative vector of order λ. The Fractional Curvature of order λ of the curve α at point r, is given by:*

$$k^\lambda(r) = \frac{k_1^\lambda(r)}{\|\alpha'(r)\|^2}, \quad \forall r \in [c,d]. \tag{52}$$

Theorem 13. *Let $\alpha : [c,d] \subset \mathbb{R} \to \mathbb{R}^2$ be a regular curve with arbitrary parameter r. Then*

$$k^\lambda(r) = \frac{a^\lambda(r)}{\|\alpha'(r)\|}k(r) + \frac{1}{\|\alpha'(r)\|^2}\frac{d}{dr}\left(b^\lambda(r)\right), \quad \forall r \in [c,d]. \tag{53}$$

Proof. By (50) and definition 5. we have

$$k^\lambda(r) = \frac{k_1^\lambda(r)}{\|\alpha'(r)\|^2} = \frac{a^\lambda(r)\|\alpha'(r)\|k(r) + \frac{d}{dr}\left(b^\lambda(r)\right)}{\|\alpha'(r)\|^2}, \quad \forall r \in [c,d].$$

Then

$$k^\lambda(r) = \frac{a^\lambda(r)}{\|\alpha'(r)\|}k(r) + \frac{1}{\|\alpha'(r)\|^2}\frac{d}{dr}\left(b^\lambda(r)\right), \quad \forall\, r \in [c,d].$$

□

Note that if the curve is parameterized by arc length, then (53) coincides with (11).

Theorem 14. *Let $\alpha : [c,d] \subset \mathbb{R} \to \mathbb{R}^2$ a regular curve with arbitrary parameter r, ${}^cD^\lambda\alpha(r) = a^\lambda(r)t(r) + b^\lambda(r)n(r)$ the fractional derivative vector. Then*

$$a^\lambda(r) = \Big[x'(r)^c D^\lambda x(r) + y'(r)^c D^\lambda y(r)\Big] \frac{1}{\|\alpha'(r)\|}, \tag{54}$$

$$b^\lambda(r) = \Big[x'(r)^c D^\lambda y(r) - y'(r)^c D^\lambda x(r)\Big] \frac{1}{\|\alpha'(r)\|}. \tag{55}$$

Proof. We have $\alpha(r) = (x(r), y(r))$ and $\alpha'(r) = (x'(r), y'(r))$. Then the Frenet-Serret frame in arbitrary parameter r is $\{t(r), n(r)\}$, where

$$t(r) = \frac{\alpha'(r)}{\|\alpha'(r)\|}, \qquad n(r) = \left(-\frac{y'(r)}{\|\alpha'(r)\|}, \frac{x'(r)}{\|\alpha'(r)\|}\right).$$

Furthermore:

$$\begin{aligned} {}^cD^\lambda\alpha(r) &= \Big({}^cD^\lambda x(r), {}^cD^\lambda y(r)\Big) = a^\lambda(r)t(r) + b^\lambda(r)n(r) \\ &= a^\lambda(r)\left(\frac{x'(r)}{\|\alpha'(r)\|}, \frac{y'(r)}{\|\alpha'(r)\|}\right) + b^\lambda(r)\left(-\frac{y'(r)}{\|\alpha'(r)\|}, \frac{x'(r)}{\|\alpha'(r)\|}\right). \end{aligned} \tag{56}$$

From (56), we get the system

$$\left\{ \begin{array}{rcl} x'(r)a^\lambda(r) - y'(r)b^\lambda(r) &=& {}^cD^\lambda x(r)\|\alpha'(r)\| \\ y'(r)a^\lambda(r) + x'(r)b^\lambda(r) &=& {}^cD^\lambda y(r)\|\alpha'(r)\| \end{array} \right. . \tag{57}$$

The determinant of the system (57) is:

$$\begin{vmatrix} x'(r) & -y'(r) \\ y'(r) & x'(r) \end{vmatrix} = \|\alpha'(r)\|^2 \neq 0, \quad \forall r \in [c,d].$$

Therefore, the system has a unique solution:

$$a^\lambda(r) = \Big[x'(r)^c D^\lambda x(r) + y'(r)^c D^\lambda y(r)\Big] \frac{1}{\|\alpha'(r)\|},$$
$$b^\lambda(r) = \Big[x'(r)^c D^\lambda y(r) - y'(r)^c D^\lambda vvx(r)\Big] \frac{1}{\|\alpha'(r)\|}.$$

□

4 Applications

In this section we present some examples where the fractional curvature of plane curves with arbitrary parameter r is calculated; and it is observed that the results give a better approximation to the integer curvature, when compared to the fractional curvature of Yajima et al. (2018), theorem 3.3, which is given by

$$k^{(\lambda)} = \left\{\frac{\Gamma(2-\lambda)}{\lambda}\right\}^{\frac{1}{\lambda}} \left[\lambda \int_0^r \sqrt{\dot{x}^2 + \dot{y}^2} dr\right]^{1-\frac{1}{\lambda}} k(r). \tag{58}$$

Example. Consider the parameterized curve $\alpha(r) = (r, \frac{r^2}{2}) = (x(r), y(r))\,, \quad \forall\ r \in \mathbb{R}$.
The integer curvature is given by

$$k(r) = \frac{1}{(1+r^2)^{3/2}}. \tag{59}$$

By (53), the fractional curvature of order λ, $0 < \lambda < 1$, given in this paper is

$$k^{\lambda}(r) = \frac{(2\lambda - \lambda^2)(r^{1-\lambda} + r^{3-\lambda})}{\Gamma(3-\lambda)(1+r^2)^{5/2}}. \tag{60}$$

Figure 1 shows the graph of $k^{\lambda}(r)$ for different values of λ : 0.7, 0.8, 0.9, 1.0. As seen in Figure 1, as λ approaches to 1, the graph of the function $k^{\lambda}(r)$ approaches the graph of the integer curvature $k(r)$.

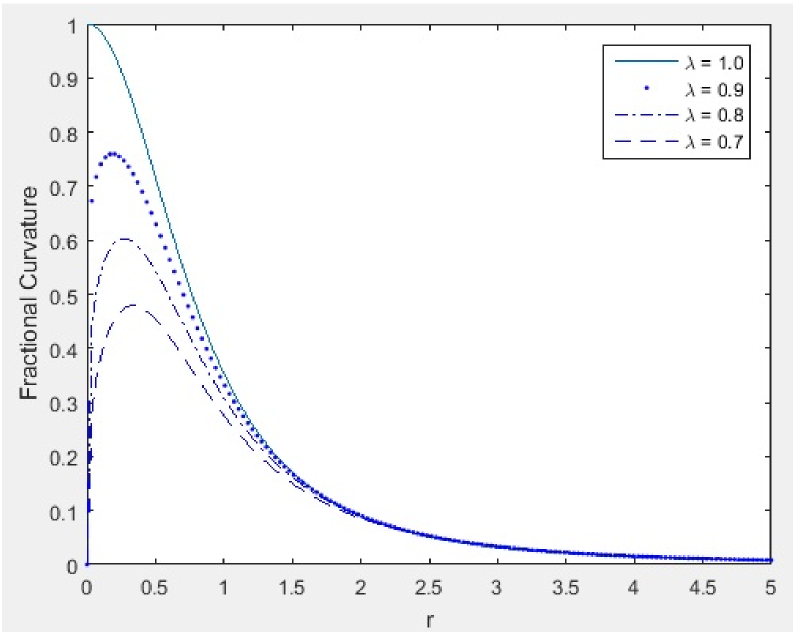

Figure 1: Graph of fractional curvature for different values of λ

The fractional curvature of Yajima et al., is given by

$$k^{(\lambda)}(r) = \frac{\lambda^{1-\frac{2}{\lambda}}\{\Gamma(2-\lambda)\}^{\frac{1}{\lambda}}}{(1+r^2)^{3/2}} \left[\frac{1}{2}\left\{r\sqrt{1+r^2} + log(r+\sqrt{1+r^2})\right\}\right]^{1-\frac{1}{\lambda}}. \tag{61}$$

Table 1 shows the results of evaluating the curvatures $k(r), k^{\lambda}(r), k^{(\lambda)}(r)$, and the errors $E^{\lambda}(r) = \|k(r) - k^{\lambda}(r)\|$, $E^{(\lambda)}(r) = \|k(r) - k^{(\lambda)}(r)\|$, for $\lambda = 0.9$, at different points of the interval $[0, 5]$.

As seen in table 1, the error $E^{\lambda}(r)$ is less than the error $E^{(\lambda)}(r)$; which indicates that the fractional curvature given in this paper gives a better approximation to the entire curvature than that provided by Yajima et al.

Table 1: Comparison between the fractional curvature $k^{\lambda}(r)$ and the fractional curvature $k^{(\lambda)}(r)$ of Yajima et al. (2018) for $\lambda = 0.9$

r	$k(r)$	$k^{\lambda}(r)$	$k^{(\lambda)}(r)$	$E^{\lambda}(r)$	$E^{(\lambda)}(r)$
0.0505	0.9962	0.6992	1.4937	0.2970	0.4975
0.1010	0.9849	0.7408	1.3671	0.2440	0.3822
0.1515	0.9665	0.7571	1.2822	0.2094	0.3157
0.2020	0.9418	0.7592	1.2096	0.1825	0.2679
0.2525	0.9114	0.7514	1.1415	0.1601	0.2301
0.3030	0.8765	0.7359	1.0753	0.1406	0.1987
0.3535	0.8381	0.7145	1.0100	0.1235	0.1720
0.4040	0.7971	0.6887	0.9458	0.1084	0.1488
0.4545	0.7545	0.6596	0.8830	0.0984	0.1285
0.5051	0.7112	0.6284	0.8220	0.0828	0.1108
0.5556	0.6680	0.5959	0.7633	0.0721	0.0953

Example. Consider the circumference $\alpha(r) = (cos(r), sin(r))$, $\forall r \in [0, 2\pi]$. The integer curvature is given by

$$k(r) = 1, \quad \forall r \in [0, 2\pi].$$

The fractional curvature $k^{\lambda}(r)$ of order λ , $0 < \lambda < 1$, is given by

$$k^{\lambda}(r) = -sin(r)S_1'(r) - cos(r)S_2'(r),$$

where

$$S_1(r) = \sum_{k=0}^{+\infty} \frac{(-1)^k r^{2k+1-\lambda}}{\Gamma(2k+2-\lambda)}, \qquad S_2(r) = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1} r^{2k+2-\lambda}}{\Gamma(2k+3-\lambda)}.$$

Figure 2 shows the graph of $k^{\lambda}(r)$ for different values of λ : 0.7, 0.8, 0.9, 1.0. As seen in Figure 2, as λ approaches to 1, the graph of the function $k^{\lambda}(r)$ approaches the graph of the integer curvature $k(r)$.

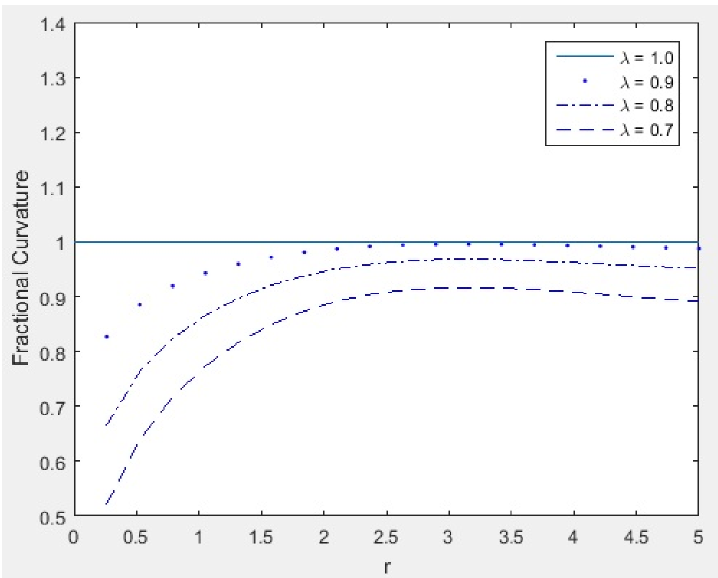

Figure 2: Graph of $k^{\lambda}(r)$, for different values of λ

The fractional curvature of Yajima et al., is given by

$$k^{(\lambda)} = \lambda^{1-\frac{2}{\lambda}} \left\{\Gamma(2-\lambda)\right\}^{\frac{1}{\lambda}} r^{1-\frac{1}{\lambda}}.$$

Table 2 shows the results of evaluating the curvatures $k(r), k^{\lambda}(r), k^{(\lambda)}(r)$, and the errors $E^{\lambda}(r) = \|k(r) - k^{\lambda}(r)\|$, $E^{(\lambda)}(r) = \|k(r) - k^{(\lambda)}(r)\|$, for $\lambda = 0.95$, at different points of the interval $[0, 5]$.

As seen in table 2, the error, the error $E^{\lambda}(r)$ is less than the error $E^{(\lambda)}(r)$; which indicates that the fractional curvature given in this paper gives a better approximation to the entire curvature than that provided by Yajima et al.

Table 2: Comparison between the fractional curvature $k^{\lambda}(r)$ and the fractional curvature $k^{(\lambda)}(r)$ of Yajima et al. (2018) for $\lambda = 0.95$

r	$k(r)$	$k^{\lambda}(r)$	$k^{(\lambda)}(r)$	$E^{\lambda}(r)$	$E^{(\lambda)}(r)$
0.1020	1.0000	0.8706	1.1602	0.1294	0.1602
0.2041	1.0000	0.9012	1.1186	0.0988	0.1186
0.3061	1.0000	0.9194	1.0950	0.0806	0.0950
0.4082	1.0000	0.9325	1.0785	0.0675	0.0785
0.5102	1.0000	0.9426	1.0659	0.0574	0.0659
0.6122	1.0000	0.9508	1.0557	0.0492	0.0557
0.7143	1.0000	0.9576	1.0472	0.0424	0.0472
0.8163	1.0000	0.9634	1.0399	0.0366	0.0399
0.9184	1.0000	0.9684	1.0335	0.0316	0.0335
1.0204	1.0000	0.9728	1.0277	0.0272	0.0277
1.1224	1.0000	0.9766	1.0226	0.0234	0.0226

Example. Consider the parameterized curve

$$\alpha(r) = \left(r, \frac{r^3}{3}\right) = (x(r), y(r)), \quad \forall r \in < 0, +\infty > .$$

The integer curvature is given by

$$k(r) = \frac{2r}{(1+r^4)^{3/2}}.$$

The fractional curvature $k^{\lambda}(r)$ of order λ , $0 < \lambda < 1$, is given by

$$k^{\lambda}(r) = (r^{2-\lambda} + r^{6-\lambda}) \frac{C^{\lambda}}{(1+r^4)^{5/2}},$$

where

$$C^{\lambda} = \frac{\lambda - 1}{\Gamma(2-\lambda)} + \frac{2(3-\lambda)}{\Gamma(4-\lambda)}.$$

Figure 3 shows the graph of $k^{\lambda}(r)$ for different values of λ : 0.7, 0.8, 0.9, 1.0.

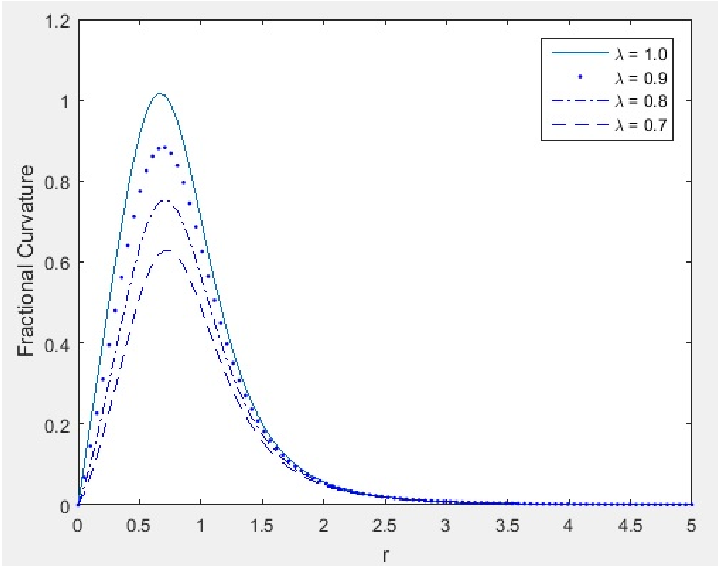

Figure 3: Graph of $k^{\lambda}(r)$, for different values of λ

Table 3 shows the fractional curvature for different values of λ : 0.7, 0.8, 0.96 and 1.00; at different points of the interval $[0, 5]$.

Table 3: Fractional curvature for different λ.

r	$k(r)$	$k^{0.96}(r)$	$k^{0.8}(r)$	$k^{0.7}(r)$
0.1020	0.2040	0.1792	0.1032	0.0710
0.2041	0.4071	0.3676	0.2366	0.1744
0.3061	0.6043	0.5546	0.3809	0.2923
0.4082	0.7835	0.7274	0.5231	0.4132
0.5102	0.9248	0.8663	0.6456	0.5215
0.6122	1.0053	0.9486	0.7279	0.5987
0.7143	1.0097	0.9586	0.7539	0.6298
0.8163	0.9408	0.8980	0.7215	0.6108
0.9184	0.8204	0.7868	0.6442	0.5518
1.0204	0.6783	0.6532	0.5439	0.4708
1.1224	0.5394	0.5215	0.4409	0.3853
1.2245	0.4183	0.4058	0.3479	0.3067
1.3265	0.3200	0.3114	0.2704	0.2403
1.4286	0.2434	0.2376	0.2088	0.1869
1.5306	0.1852	0.1813	0.1611	0.1452
1.6327	0.1415	0.1389	0.1247	0.1131
1.7347	0.1088	0.1070	0.0970	0.0886
1.8367	0.0843	0.0831	0.0761	0.0698
1.9388	0.0659	0.0651	0.0601	0.0555
2.0408	0.0519	0.0514	0.0478	0.0444
2.1429	0.0413	0.0410	0.0384	0.0358
2.2449	0.0331	0.0329	0.0311	0.0291

4.1 Example of a block moving alon a parabolic arc

This example is motivated by Lapidus (1984) paper, and it will be shown that the magnitude of the total velocity of a block moving along a parabolic arc depends on the fractional curvature of the trajectory.

Consider the motion of a block of mass m moving on the parabolic arc $y = \frac{x^2}{2}, 0.1 \leq x \leq 3$. The block starts from rest at the point $x = 3$ and moves along the parabolic arc, acting on it the gravity g and the kinetic sliding friction μ.

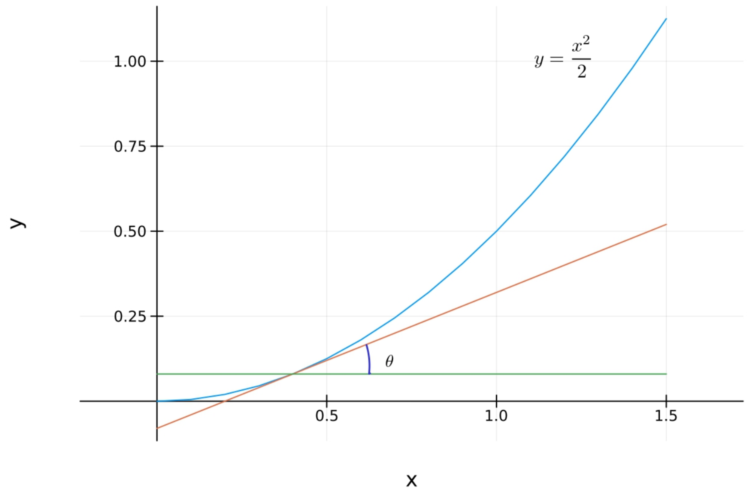

Figure 4: Displacement path

The trajectory is parameterized by $\alpha(r) = \left(r, \frac{r^2}{2}\right) = (x(r), y(r))\,, \quad r \in [0.1, 3]$.

The equations of motion, Lapidus (1984), are of the form:

$$\begin{aligned} m\ddot{x} &= -N\sin\theta + f\cos\theta, \\ m\ddot{y} &= mg + N\cos\theta + f\sin\theta, \end{aligned}$$

where

- θ is the angle with the horizontal,
- $f = \pm\mu N$,
- μ is the coefficient of kinetic friction.

The sign of f is the opposite of the direction of motion of the block.

By integrating we obtain

$$\ln\left(1 + \frac{V_x}{g}\right) = 2\ln\left(\frac{\cos\theta}{\cos\theta_0}\right) \pm 2\mu\left(\theta - \theta_0\right),$$

where

- θ_0 is the initial angle of the block,
- V_x is the horizontal velocity of the block.

The total velocity is given by

$$V = V_x\cos\theta.$$

In addition, as the derivitative

$$y'(x) = x = \tan\theta,$$

the integer curvature of α, is

$$k = \cos^3\theta.$$

Therefore, the horizontal velocity as a function of the integer curvature is given by

$$V_x = \sqrt{g\left[\left(\frac{k}{k_0}\right)^{2/3} e^{2\mu\left(\arccos\left(k^{\frac{1}{3}}\right)-\arccos\left(k_0^{\frac{1}{3}}\right)\right)}-1\right]},$$

and the magnitude of the total velocity in terms of the integer curvature is given by

$$V = \frac{V_x}{k^{1/3}}.$$

Using the fractional curvature given at this paper (60), the horizontal velocity as a function of the fractional curvature of order λ, is given by

$$V_{x,\lambda} = \sqrt{g\left[\left(\frac{k^\lambda}{k_0^\lambda}\right)^{2/3} e^{2\mu\left(\arccos\left((k^\lambda)^{\frac{1}{3}}\right)-\arccos\left((k_0^\lambda)^{\frac{1}{3}}\right)\right)}-1\right]},$$

and the magnitude of the total fractional Velocity of order λ, is given by

$$V^\lambda = \frac{V_{x,\lambda}}{(k^\lambda)^{1/3}}.$$

In this example we consider $\mu = 0.2, g = 9.8$, and using a partition with 30 points in the interval $[0.1; 3]$, the data in Table 4 are obtained for different values of $\lambda : 0.7, 0.8, 0.9$ y 1.0.

The r_i values denote the partition points, which are sorted in descending order to be able to analyze the results.

Thus, for $k^{0.7} = 0.0343$, $k^{0.8} = 0.0343$, $k^{0.9} = 0.0334$ and $k^{1.0} = 0.0316$, the fractional total velocity and integer total velocity are identically zero; this indicates that the block is at rest.

Furthermore, it is observed that $k^{0.7}$ and $k^{0.8}$ grow to 0.4776 and 0.6018 respectively, and $V^{0.7}$ and $V^{0.8}$ grow to 6.9385 and 7.3801respectively, and $k^{0.7}$ and $k^{0.8}$ decrease to 0.3851 and 0.5416 respectively, and also $V^{0.7}$ amd $V^{0.8}$ decrease to 6.5601 and 7.1526 respectively.

Finally, $k^{0.9}$ grow to 0.7594 and decrease to 0.7403, $V^{0.9}$ grow to 7.7811 and decrease to 7.7426.

It is observed that these decreases in the values occur near the left end of the interval, as shown in Table 4.

Table 4: Total Fractional Velocity for different values of the fractional curvature of order λ

r_i	$k^{0.7}$	$V^{0.7}$	$k^{0.8}$	$V^{0.8}$	$k^{0.9}$	$V^{0.9}$	$k^{1.0} = k$	$V^{1.0} = V$
3.0000	0.0343	0	0.0343	0	0.0334	0	0.0316	0
2.9000	0.0372	1.0441	0.0373	1.0658	0.0365	1.0907	0.0346	1.1189
2.8000	0.0404	1.4924	0.0407	1.5238	0.0399	1.5598	0.0380	1.6005
2.7000	0.0440	1.8479	0.0445	1.8872	0.0438	1.9324	0.0419	1.9834
2.6000	0.0481	2.1578	0.0488	2.2045	0.0482	2.2579	0.0463	2.3183
2.5000	0.0526	2.4405	0.0536	2.4941	0.0531	2.5554	0.0512	2.6247
2.4000	0.0577	2.7053	0.0591	2.7657	0.0587	2.8348	0.0569	2.9127
2.3000	0.0635	2.9579	0.0652	3.0251	0.0652	3.1019	0.0634	3.1884
2.2000	0.0700	3.2019	0.0723	3.2761	0.0725	3.3607	0.0709	3.4560
2.1000	0.0774	3.4400	0.0803	3.5213	0.0810	3.6140	0.0795	3.7183
2.0000	0.0859	3.6741	0.0895	3.7629	0.0907	3.8638	0.0894	3.9774
1.9000	0.0955	3.9056	0.1001	4.0021	0.1019	4.1119	0.1010	4.2352
1.8000	0.1066	4.1356	0.1122	4.2404	0.1149	4.3594	0.1145	4.4929
1.7000	0.1192	4.3650	0.1263	4.4785	0.1300	4.6073	0.1303	4.7517
1.6000	0.1337	4.5942	0.1425	4.7172	0.1476	4.8564	0.1489	5.0125

1.5000	0.1503	4.8237	0.1613	4.9568	0.1681	5.1072	0.1707	5.2757
1.4000	0.1694	5.0533	0.1830	5.1974	0.1921	5.3598	0.1964	5.5418
1.3000	0.1913	5.2828	0.2081	5.4387	0.2201	5.6143	0.2267	5.8107
1.2000	0.2161	5.5113	0.2371	5.6802	0.2528	5.8699	0.2624	6.0820
1.1000	0.2443	5.7375	0.2703	5.9205	0.2907	6.1257	0.3044	6.3548
1.0000	0.2758	5.9592	0.3081	6.1577	0.3345	6.3797	0.3536	6.6273
0.9000	0.3103	6.1734	0.3503	6.3890	0.3844	6.6294	0.4107	6.8970
0.8000	0.3473	6.3759	0.3968	6.6105	0.4405	6.8709	0.4761	7.1600
0.7000	0.3853	6.5611	0.4461	6.8170	0.5019	7.0991	0.5498	7.4112
0.6000	0.4219	6.7215	0.4960	7.0016	0.5668	7.3077	0.6305	7.6439
0.5000	0.4533	6.8475	0.5427	7.1561	0.6316	7.4891	0.7155	7.8499
0.4000	0.4743	6.9262	0.5806	7.2702	0.6909	7.6347	0.8004	8.0197
0.3000	0.4776	6.9385	0.6018	7.3301	0.7370	7.7358	0.8787	8.1434
0.2000	0.4538	6.8493	0.5954	7.3124	0.7594	7.7811	0.9429	8.2117
0.1000	0.3851	6.5601	0.5416	7.1526	0.7403	7.7426	0.9852	8.2175

Using the data from Table 4, the total fractional and integer velocity is plotted as a function of the fractional and integer curvature of the trajectory for different values of $\lambda : 0.7, 0.8, 0.9$ and 1.0.

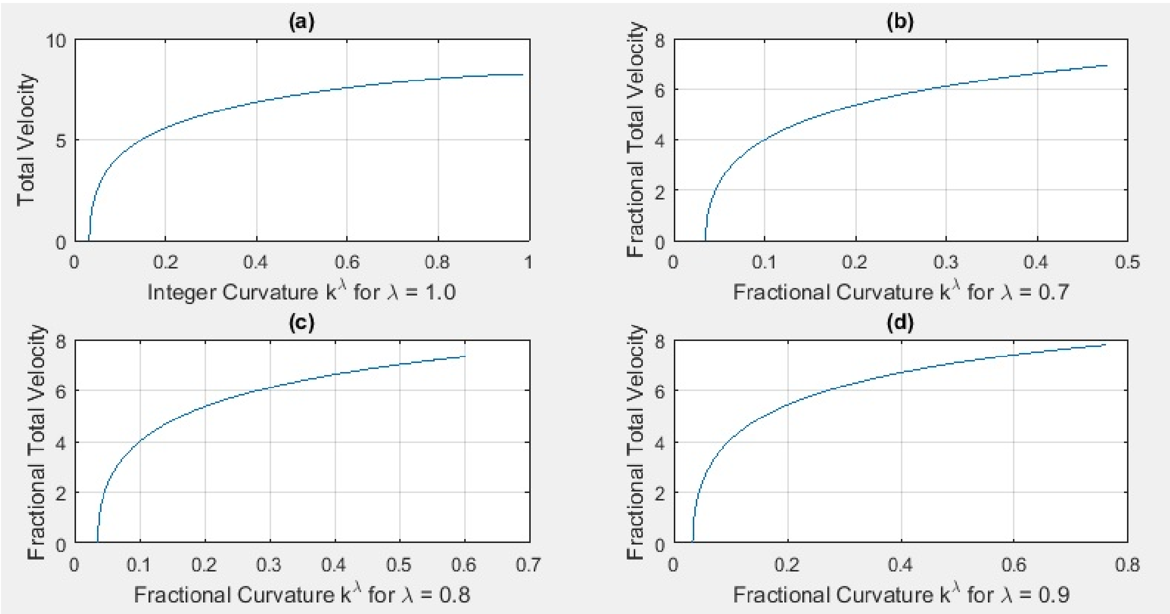

Figure 5: Fractional Total Velocity vs Fractional Curvature

Figure 5(a) shows the behavior of the integer total velocity as a function of the integer curvature of the trajectory. Figures 5(b), 5(c) and 5(d) show the behavior of the total fractional velocity as a function of the fractional curvature of the trajectory; and it is observed that when λ is close to 1, the curve approaches the integer case.

5 Conclusions

In this paper a new definition of Fractional Curvature of plane curves was introduced, using Caputo's fractional derivative of order λ $(0 < \lambda < 1)$. The importance of our study lies in the analysis of the geometrical properties of the curve from the point of view of fractional calculus; thus it has been proved that the fractional curvature belongs to the intrinsic geometry of the curve; since it is invariant under isometries. The 1-dimensional Euclidean spaces were also characterized as those curves whose fractional curvature of order λ is zero at all its points for all λ. In addition, an example showed the relationship between the velocity of a body moving on a parabolic arc, and the fractional curvature of the parabolic arc.

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